MathPath at Lewis & Clark College

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Visual Number Theory

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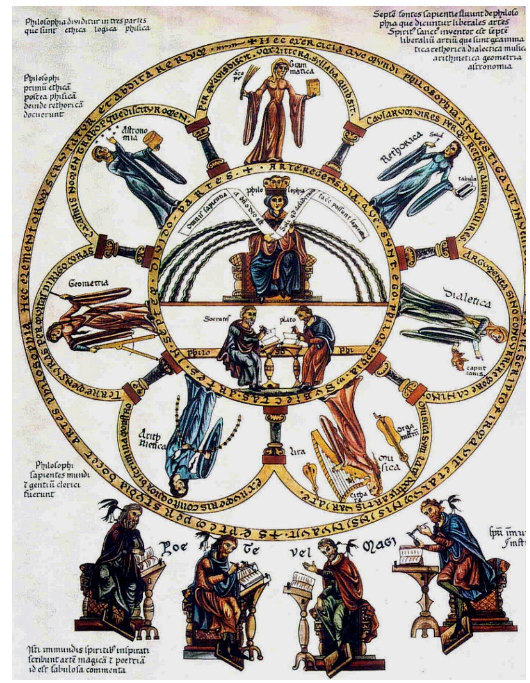
Introduction

From the website www.lclark.edu: "*The mission of Lewis & Clark is to know the traditions of the liberal arts, to test their boundaries through ongoing exploration, and to hand on to successive generations the tools and discoveries of this quest.*"

What are the liberal arts? Here's a quote from the *Encyclopædia Britannica*: a "college or university curriculum aimed at imparting general knowledge and developing general intellectual capacities in contrast to a professional, vocational, or technical curriculum. In the medieval European university the seven liberal arts were grammar, rhetoric, and logic (the *trivium*) and geometry, arithmetic, music, and astronomy (the *quadrivium*)." At the right is an engraving from 1180 entitled "Philosophy reigning over the seven liberal arts."



One of the seven seems out of place—arithmetic. But until the 20th century, "arithmetic" referred to the branch of mathematics we now call number theory, the branch of mathematics devoted to the study of the integers.

While preparing this talk, I looked through several elementary number theory texts (in the context of number theory, "elementary" means no use of complex analysis), I was struck by how few illustrations most of them have.

A number can represent many things—the cardinality of a set, the length of a line segment, the area of a region, the volume of a solid, etc. The texts should have more pictures.

In this talk I'll present some of my favorite pictures for use in studying number theory, what I call *visual gems of number theory*.

A brief outline:

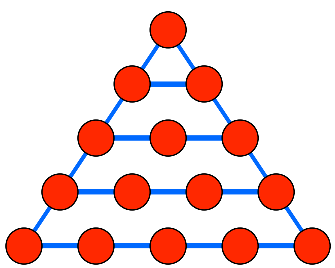
1. Figurate numbers
2. Patterns among polygonal numbers
3. Triangular numbers and binomial coefficients
4. Congruence
5. Pythagorean triples
6. The carpets theorem
7. Rational and irrational numbers
8. Perfect numbers

Figurate Numbers

The *figurate numbers* are positive integers that can be represented by geometric patterns (thus the words *squares* and *cubes* for certain numbers). The study of figurate numbers dates back to the time of Pythagoras and his colleagues.

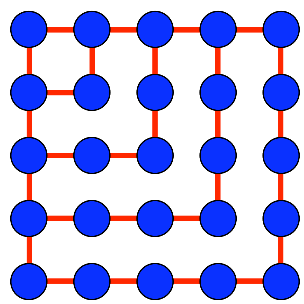
Special cases include the *polygonal numbers*, which can be represented by sets of points in the plane. Here are some examples:

Triangular numbers



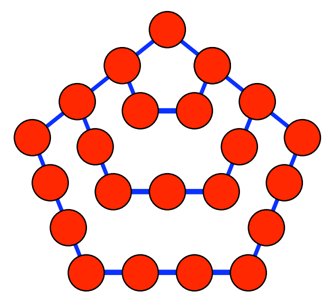
$t_n : 1, 3, 6, 10, 15, \ldots$

Square numbers



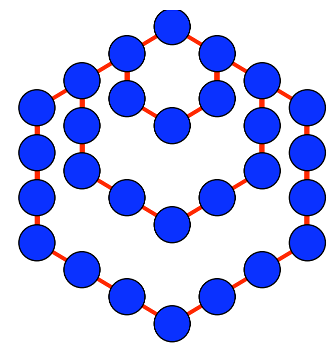
$s_n : 1, 4, 9, 16, 25, \ldots$

Pentagonal numbers



$p_n : 1, 5, 12, 22, \ldots$

Hexagonal numbers



$h_n : 1, 6, 15, 28, \ldots$

To compute t_{100} or p_{200} would be tedious—are there formulas for these numbers?

Let's examine the data and look for patterns: The entries in the table are the nth k-gonal numbers for $3 \le k \le 8$ and $1 \le n \le 6$:

$n \backslash k$	3	4	5	6	7	8
1	1	1	1	1	1	1
2	3	4	5	6	7	8
3	6	9	12	15	18	21
4	10	16	22	28	34	40
5	15	25	35	45	55	65
6	21	36	51	66	81	96

The data appear to be (1) linear in k (constant differences) and (2) quadratic in n (the differences increase linearly).

Hypothesis: $n+\frac{1}{2}n(n-1)(k-2)$.

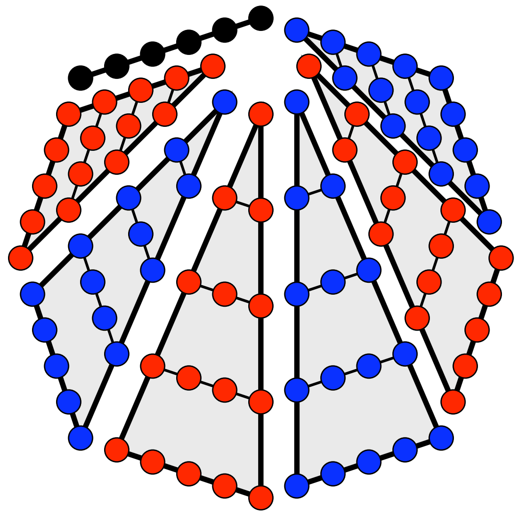
Proof: Induction? on n? on k?

> Proofs really aren't there to convince you that something is true—they're there to show you why it is true.
>
> —Andrew Gleason

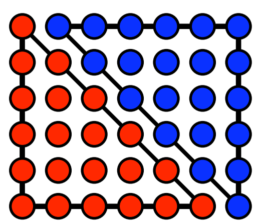
> A good proof is one that makes us wiser.
>
> —Yu. I. Manin

Let's exploit the geometry to find an expression for the nth k-gonal number:

Here is the *n*th *k*-gonal number (really the 6th octagonal number, 96):



To evaluate, we "triangulate:" Thus the *n*th *k*-gonal number is $n+(k-2)t_{n-1}$. And t_n is easy to compute:



$$2t_n = n(n+1)$$

$$\therefore t_n = \frac{n(n+1)}{2}$$

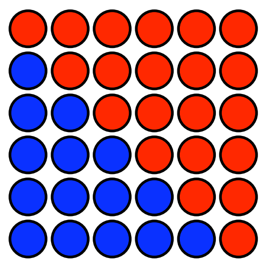
Hence the *n*th *k*-gonal number is $n+(k-2)\dfrac{n(n-1)}{2}$ $(n \geq 1, k \geq 3)$.

Relationships among the polygonal numbers

There are many lovely relationships among these numbers. Consider first the triangular and square numbers:

n	1	2	3	4	5	6	7	8	9	10
n^2	1	4	9	16	25	36	49	64	81	100
t_n	1	3	6	10	15	21	28	36	45	55

Observe that 3 + 6 = 9, 15 + 21 = 36, 36 + 45 = 81, and so on. The pattern?



$$t_{n-1} + t_n = n^2$$

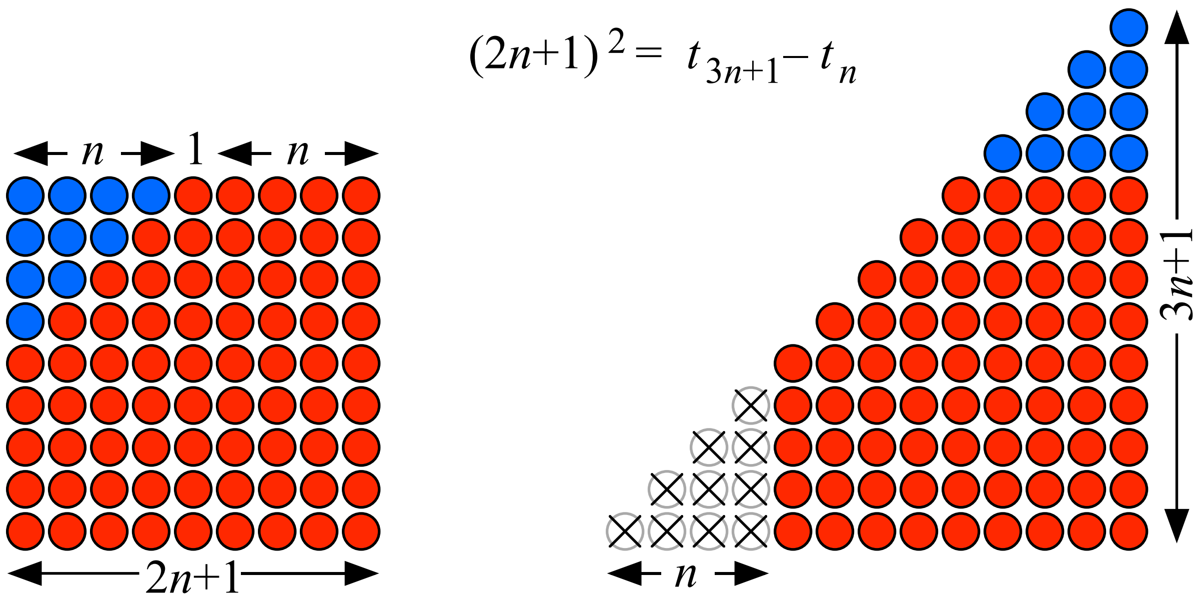
Exercise: Show that $t_n^2 + t_{n-1}^2 = t_{n^2}$ and $t_n^2 - t_{n-1}^2 = n^3$.

Now consider the triangular numbers and odd squares:

n	1	2	3	4	5	6	7	8	9	10
$(2n+1)^2$	9	25	49	81	121	169	225	289	361	441
t_n	1	3	6	10	15	21	28	36	45	55

Observe that $9 = 10 - 1$; $25 = 28 - 3$; $49 = 55 - 6$, and so on. The pattern?

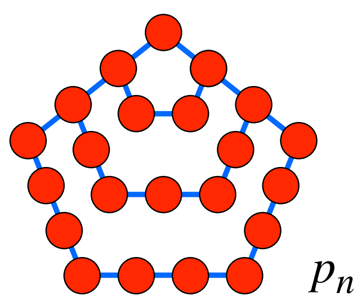
$$(2n+1)^2 = t_{3n+1} - t_n$$

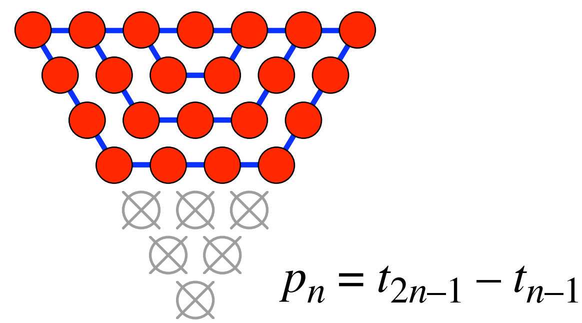


Now consider the triangular and pentagonal numbers:

n	1	2	3	4	5	6	7	8	9	10
t_n	1	3	6	10	15	21	28	36	45	55
p_n	1	5	12	22	35	51	70	92	115	145

Observe that $12 = 15 - 3$; $22 = 28 - 6$; and $35 = 45 - 10$. The pattern?

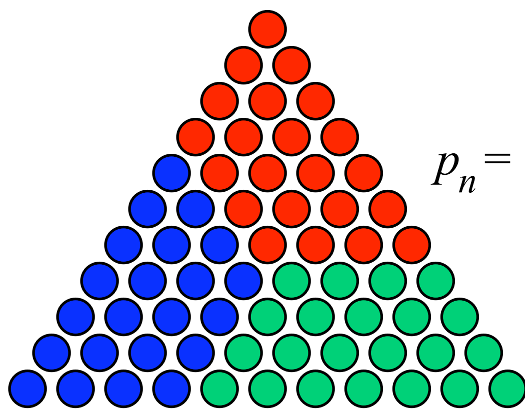




Here's another:

n	1	2	3	4	5	6	7	8	9	10
t_n	1	3	6	10	15	21	28	36	45	55
p_n	1	5	12	22	35	51	70	92	115	145

Do you see the pattern? Yes, you do:



$p_n = \frac{1}{3} t_{3n-1}$

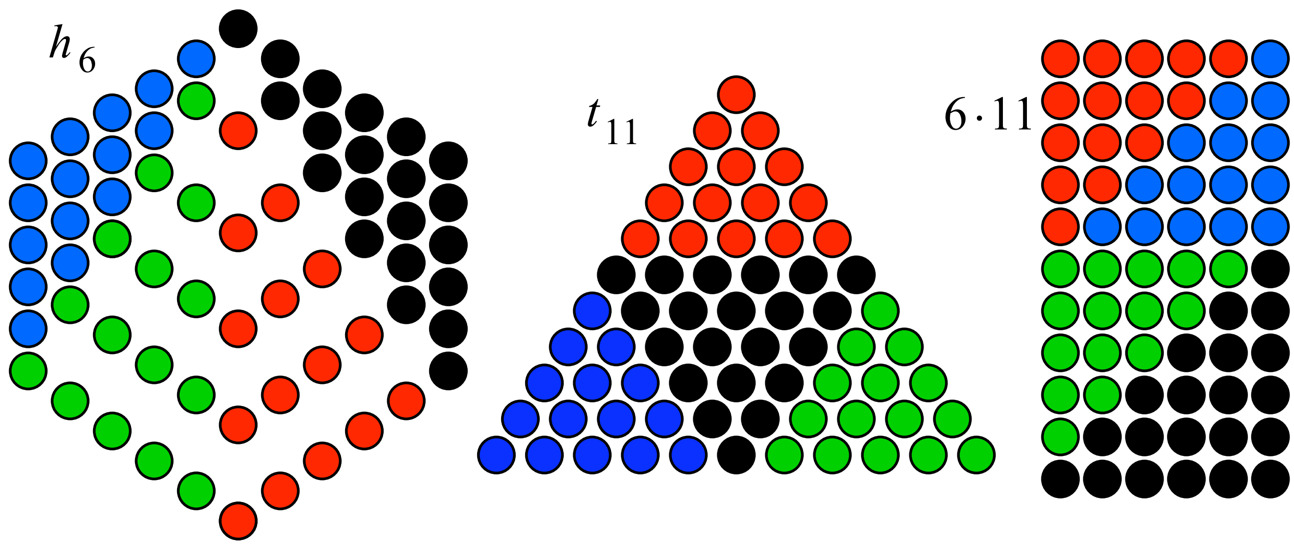
Exercise: $1 + 5 + 12 + \cdots + p_n = n \cdot t_n$.

Now consider the triangular and hexagonal numbers:

n	1	2	3	4	5	6	7	8	9	10
t_n	1	3	6	10	15	21	28	36	45	55
h_n	1	6	15	28	45	66	91	120	153	190

Is every hexagonal number a triangular number?

Yes—in fact, we have $h_n = t_{2n-1} = n(2n-1)$:

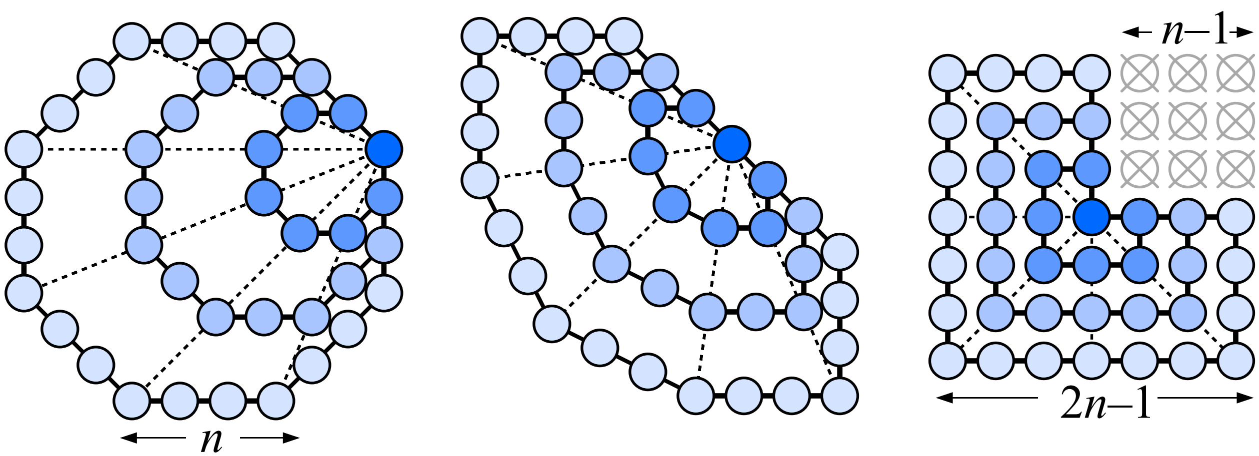


[The middle picture also shows that $t_{2n-1} = t_n + 3t_{n-1}$.]

Octagonal numbers and squares:

n	1	2	3	4	5	6	7	8	9
n^2	1	4	9	16	25	36	49	64	81
o_n	1	8	21	40	65	96	133	176	225

Here we have: $21 = 25 - 4$; $40 = 49 - 9$; and $65 = 81 - 16$. The pattern:



$$o_n = (2n-1)^2 - (n-1)^2$$

Exercise: $1 + 8 + 21 + \cdots + o_n = (2n-1)t_n$.

The triangular numbers appear in Pascal's triangle:

$$t_n = 1+2+3+\cdots+n = \binom{n+1}{2}.$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

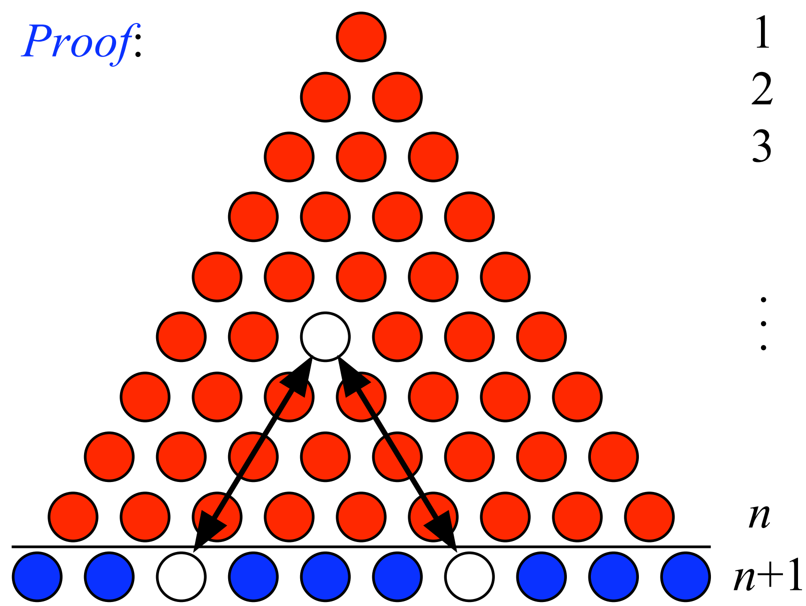
Why are triangular numbers binomial coefficients?

One answer: each equals $\dfrac{n(n+1)}{2}$.

A better answer:

There exists a one-to-one correspondence between a set of $t_n = 1 + 2 + \cdots + n$ elements and the set of *two-element subsets* of a set with $n + 1$ elements.

Proof:

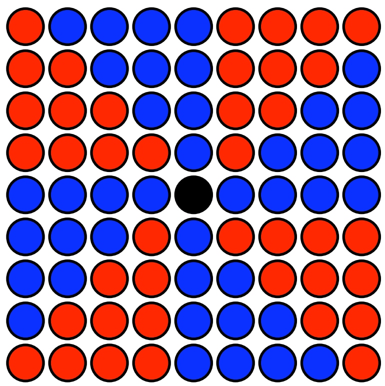


Congruence

A central concept in number theory is the notion of *congruence*: we say that

a is congruent to b modulo m [$a \equiv b \pmod{m}$] if and only if m divides $a - b$, $m > 0$.

Examples: $20 \equiv 6 \pmod{7}$, $19 \equiv -3 \pmod{11}$, etc. An exercise in nearly every number theory text: Consider the odd squares: $\{1,9,25,49,81,\ldots\}$. Each is one more than a multiple of 8, so: Prove: If n is odd, then $n^2 \equiv 1 \pmod{8}$.



$$(2k+1)^2 = 8t_k + 1$$

A consequence: *There are infinitely many square triangular numbers*.

$$t_1 = 1^2 \text{ and } t_{8t_k} = \frac{8t_k(8t_k+1)}{2} = 4t_k(2k+1)^2,$$

so if t_k is square, so is t_{8t_k}. Examples: $t_8 = 6^2$, $t_{288} = 204^2$, etc.

But not all square t_ks are generated this way: $t_{49} = 35^2$, $t_{1681} = 1189^2$, etc.

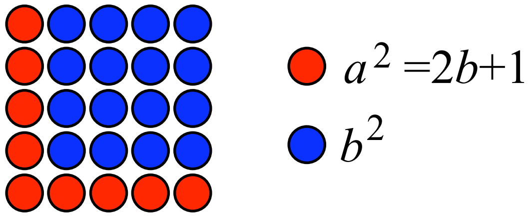
For students of number theory: Finding *all* the square triangular numbers requires solving the equation $\frac{n(n+1)}{2} = y^2$ for positive integers n and y [a *diophantine equation*, named for Diophantus of Alexandria (3rd century CE)]. This particular one is equivalent to $x^2 - 8y^2 = 1$ for $x = 2n+1$, an example of a so-called *Pell's equation*. This equation has infinitely many solutions in positive integers, and in your number theory course you will learn how to find them all.

While reading his copy of Diophantus' book *Arithmetica*, Pierre de Fermat (1601-1665) wrote that had shown that the diophantine equation $a^n + b^n = c^n$ has no solutions in integers for $n \geq 3$ (the *Last Theorem*, proved by Andrew Wiles in 1995). But what about $a^2 + b^2 = c^2$?

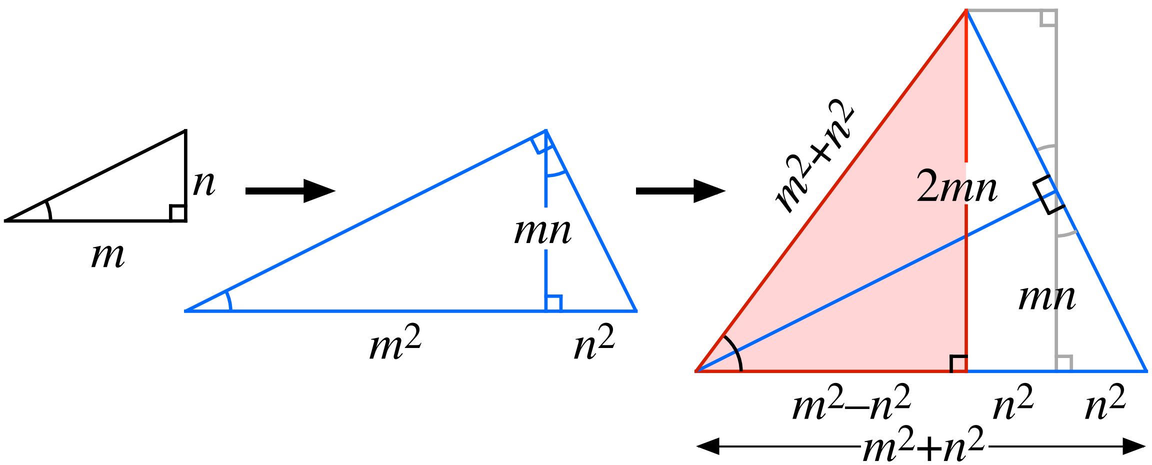
Solutions to $a^2 + b^2 = c^2$ in positive integers are called *Pythagorean triples*, thanks to the Pythagorean theorem from plane geometry. Let's examine them.

Pythagorean triples: Solutions to $a^2 + b^2 = c^2$ in positive integers. Familiar examples are $(a,b,c) = (3,4,5)$, $(5,12,13)$, $(8,15,17)$, etc. In fact, there are infinitely many:

If a is odd, so is $a^2 = 2b+1$, and $a^2 + b^2 = (b+1)^2$:



But this doesn't generate all the triples. Euclid's formula does: If (a,b,c) is a Pythagorean triple (with a even), then there exist positive integers m and n with $m > n$ such that



$$a = 2mn$$
$$b = m^2 - n^2$$
$$c = m^2 + n^2$$

Euclid also showed that the formula generates a *primitive* triple (*a, b, c* have no common factor) when $m \not\equiv n \pmod 2$ and $(m,n) = 1$.

A second characterization of Pythagorean triples:

If you can factor even squares, you can find all the Pythagorean triples, since

There is a one-to-one correspondence between Pythagorean triples (a,b,c) *and factorizations of even squares of the form* $n^2 = 2pq$.

The correspondence is $(a,b,c) = (n+p, n+q, n+p+q)$. Here's an example.

Consider factorizations of $6^2 = 36$:

$6^2 = 2\cdot 1\cdot 18$ corresponds to $(7,24,25)$;

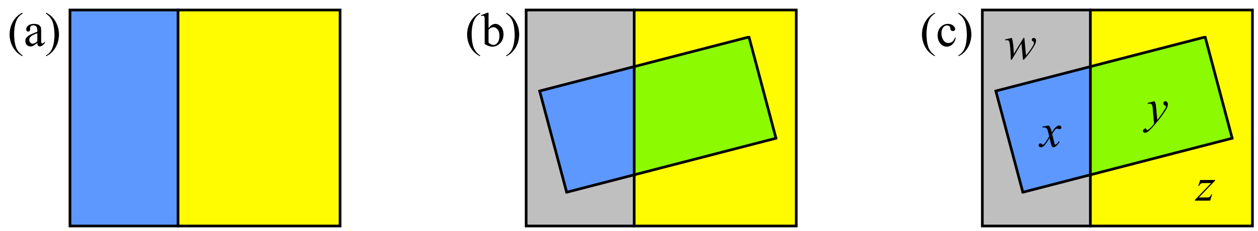
$6^2 = 2\cdot 2\cdot 9$ corresponds to $(8,15,17)$; and

$6^2 = 2\cdot 3\cdot 6$ corresponds to $(9,12,15)$ [the first two are primitive, this one isn't].

We derive this characterization with a little known theorem: *The carpets theorem*.

The carpets theorem.

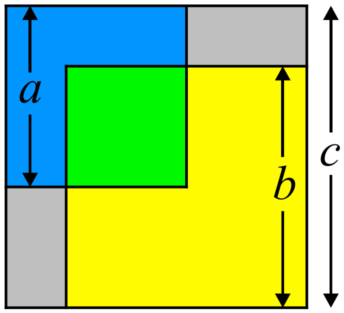
A simple but powerful result for deriving some nice results in number theory. Suppose we have a room with two carpets that completely cover the floor, as illustrated in (a). If we move one of them, as in (b), then the area of the overlap (in green) must equal the uncovered area (in gray). This is easily verified with simple algebra.



In (c), let x, y, z, and w denote the areas of the differently colored regions in the room. The area of the room is $x+y+z+w$, the combined area of the two carpets is $x+2y+z$, and $x+y+z+w = x+2y+z$ if and only if $y=w$.

Thus we have proved: *Place two carpets in a room. The area of the overlap equals the area of the uncovered floor if and only if the combined area of the carpets equals the area of the floor*. [The shapes of the room and the carpets are arbitrary.]

Now let's apply this theorem to Pythagorean triples:



The Pythagorean relation $a^2+b^2=c^2$ suggests placing carpets with areas a^2 and b^2 in a room with area c^2, as shown at the right.

By the carpets theorem the area $(a+b-c)^2$ of the green square equals the sum $2(c-a)(c-b)$ of the areas of the two gray rectangles.

Now set $n=a+b-c$, $p=c-b$, and $q=c-a$. Since n, p, and q are positive integers and $a^2+b^2=c^2$ if and only if $n^2=2pq$, we have the characterization several slides back:

There is a one-to-one correspondence between Pythagorean triples (a,b,c) and factorizations of even squares of the form $n^2 = 2pq$. [$a=n+p$, $b=n+q$, and $c=n+p+q$.]

$10^2=2\cdot1\cdot50$ corresponds to $(11,60,61)$;

$10^2=2\cdot2\cdot25$ corresponds to $(12,35,37)$; and

$10^2=2\cdot5\cdot10$ corresponds to $(15,20,25)$ [the first two are primitive, this one isn't].

Exercise:

$2^2+3^2+6^2=7^2$, so *Pythagorean quadruples* exist! Are there more? How many?

***Pythagorean Runs—Pythagorean (2k+1)-tuples*:**

$$3^2 + 4^2 = 5^2$$

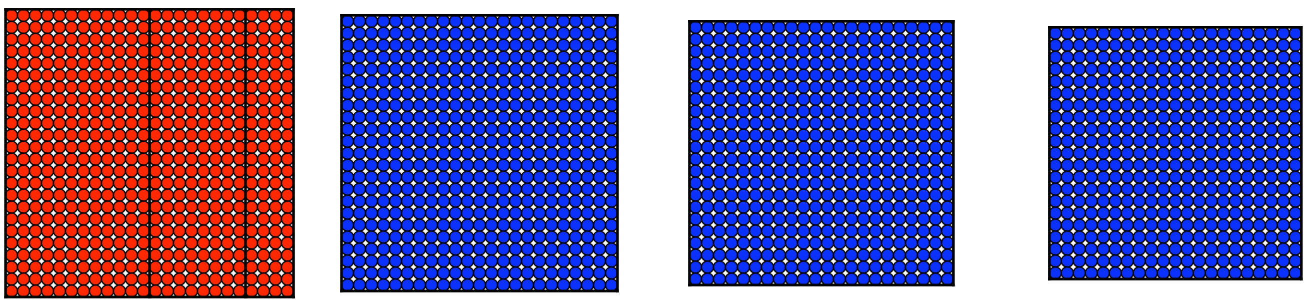
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

$$\vdots$$

A visual proof:

For $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$, others are similar:



24^2 23^2 22^2 21^2

Pythagorean Runs—Pythagorean **(2*k*+1)*-tuples*:**

$$3^2 + 4^2 = 5^2$$

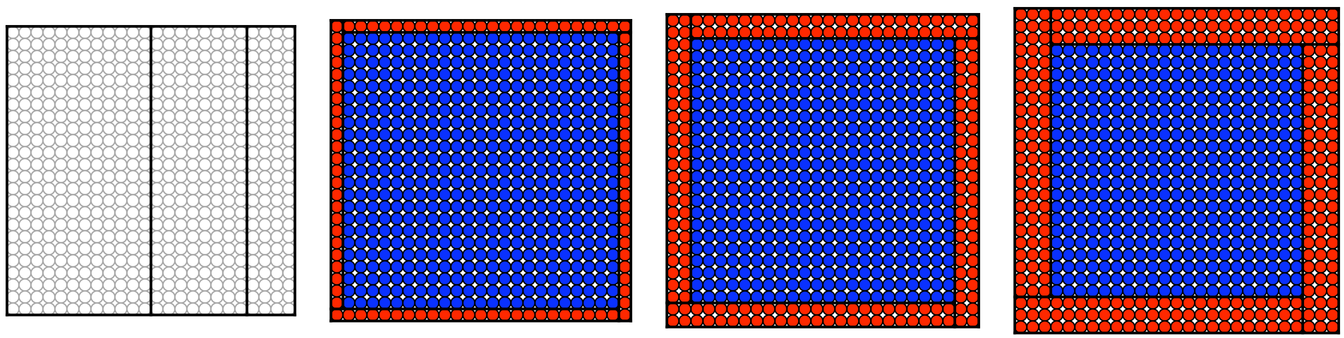
$$10^2 + 11^2 + 12^2 = 13^2 + 14^2$$

$$21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$$

$$\vdots$$

A visual proof:

For $21^2 + 22^2 + 23^2 + 24^2 = 25^2 + 26^2 + 27^2$, others are similar:



25^2 26^2 27^2

Rational and irrational numbers

In addition to properties of the integers, number theory studies objects constructed from integers, such as rational numbers (*ratios* of integers) and irrational numbers.

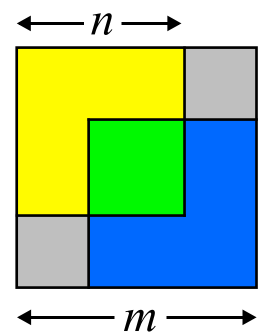
The existence of irrationals is often demonstrated by showing that $\sqrt{2}$ is irrational.

The definition of irrational (not rational) practically demands a proof by contradiction.

Assume $\sqrt{2} = m/n$, where m and n are positive integers and the fraction is in lowest terms. Then $m^2 = 2n^2$, and m and n are the smallest positive integers with this property.

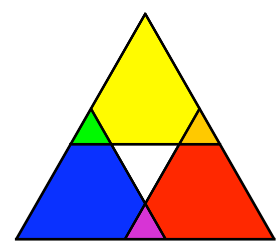
Now $m^2 = n^2 + n^2$ suggests using the carpets theorem.

Place the two carpets with area n^2 in a room with area m^2 as shown at the right. By the carpets theorem, the area of the green square equals the sum of the areas of the two gray squares. But these squares also have integer sides $2n - m$ and $m - n$ smaller than m and n, respectively (since $1 < \sqrt{2} < 2$), a contradiction. Hence $\sqrt{2}$ is irrational.



Exercise: Prove that $\sqrt{3}$ is irrational.

[Hint: Place three equilateral triangular carpets in an equilateral triangular room, as shown at the right.]



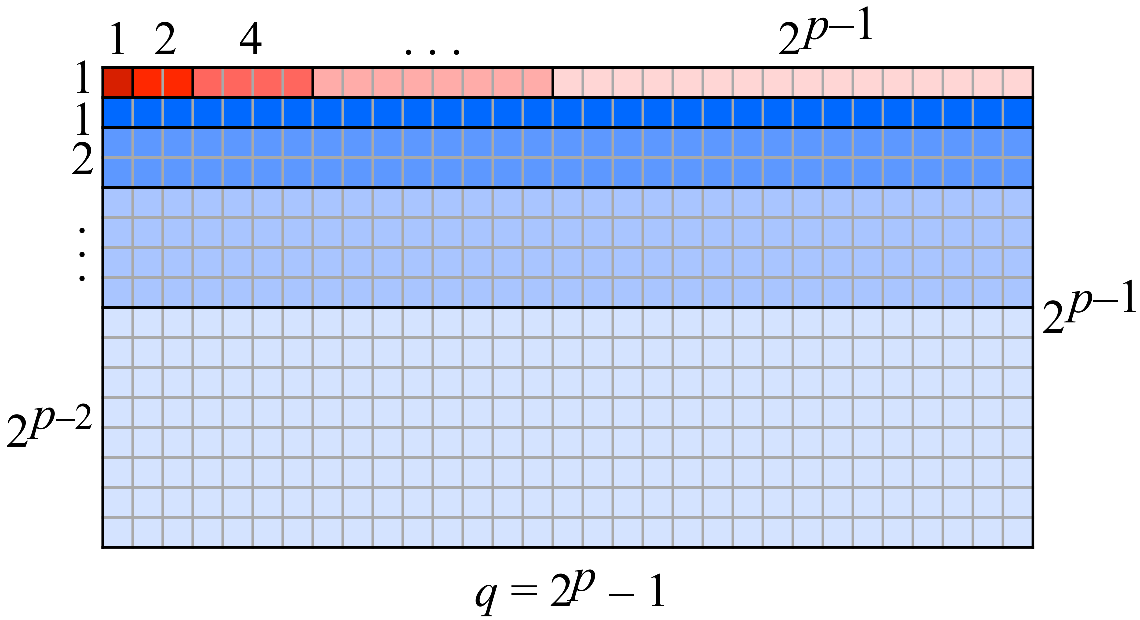
Perfect numbers

A *perfect number* is one that is equal to the sum of its *proper divisors* (a divisor is proper if it is less than the number). The proper divisors of 6 are 1, 2, and 3; and $6 = 1 + 2 + 3$. The proper divisors of 28 are 1, 2, 4, 7, and 14; and $28 = 1 + 2 + 4 + 7 + 14$.

Euclid showed us how to find *even* perfect numbers:

If p and $q = 2^p - 1$ are prime, then $N = 2^{p-1}q$ is perfect.

The proper divisors of $N = 2^{p-1}q$ are $1, 2, 4, \cdots, 2^{p-1}, q, 2q, 4q, \cdots, 2^{p-2}q$; their sum is N:



Leonhard Euler showed that any even perfect number must have this form.

Final exercise: Show that every even perfect number is a triangular number.

THANK YOU!

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